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An Algorithm for calculating the Lyapunov Exponents of Impulsive Switched Systems and its Application

Guodong Liu, Shiqi Xiong, Qingdu Li[a, *]

Research Center of Analysis and Control for Complex Systems, Chongqing University of Posts and Telecommunications, Chongqing, 400065, China

[a]liqd@outlook.com

Abstract: Lyapunov exponents are important values for the characterization of finite-dimensional nonlinear dynamic systems. But most of the generalized algorithm are unfit for impulsive switched systems. This paper analyzes the evolution of adjacent trajectories over the impulsive switched planes, and at the same time, use phase space geometric class rules to obtain Jacobian matrices of the alteration adjacent trajectories before and after the switches.

Keywords: Chaos, Impulse Switched Systems, Lyapunov Exponents, Dynamic Walking

1. Introduction

In recent years, some scholars have been proposed several methods to calculate the Lyapunov[1-5] exponent for a specific switching system. Stefanski et al. using the synchronization between two systems to estimate the maximum Lyapunov exponent[6-9]. Galvanetto using discrete mapping algorithm to calculate the Lyapunov exponent of the low dimensional stick-slip mechanical system. De Souza and Caldas adding a set of conversion conditions in the collision for two specific mechanical systems with collision phenomena, and then use the algorithm of Lyapunov exponent for the smooth discrete dynamical system to calculate Lyapunov exponent.

This paper through the phase space geometry deduced a Jacobian matrix of the trajectory's change in pulse plane. And use the matrix to compensate Lyapunov exponent which is discontinuousness in switching plane. Calculating n Lyapunov exponents of the *n*-dimensional pulse switching system.

2. Algorithm derivation of pulse switched system LE

Impulsive switched system (1) is composed of multiple consecutive subsystem and correspondingly impulse function. Consider the equation of state of impulsive switched system:

$$\begin{cases} \dot{x} = f_\lambda(x), x \in \mathrm{D}_\lambda, \lambda = i \\ S_j = 0, x^+ = h_{ij}(x^-), \lambda = j \\ x_0 = x^+, j \neq i, \lambda \in \{1, 2, \cdots, k\} \end{cases} \tag{1}$$

define in an n-dimensional state space where contains the number of continuous subsystem with n-dimensional is k. $D_\lambda(x)$ is the state space of subsystem $f_\lambda(\mathrm{x})$. $S_j = 0$ is a switching condition when subsystem *i* switched to j by impulsive function $x^+ = h_{ij}(x^-),\ \ x^- \in D_i(x)$, $x^+ \in D_j(x)$. Where $f_j(x)$ with initial condition x_0.

In view of the system characteristics, it can be divided into two parts respectively to analyze that continuous subsystems and the impulsive switch.

2.1 calculating the Lyapunov exponents of continuous subsystems

Usually, the calculation of the exponents is performed considering the observed trajectory $x(t)$ as a solution of the continuously differentiable dynamical subsystem

$$\dot{\mathbf{x}} = \mathbf{f}_\lambda(\mathbf{x}(t)),\ \ \mathbf{x}(t_0) = \mathbf{x}_0,\ \ \mathbf{f}_\lambda \in C,\ \ \lambda \in 1, 2 \cdots k \tag{2}$$

Where k is the number of subsystem. Defined in an *n*-dimensional state space where $\mathbf{f}_\lambda$ is a continuously differentiable vector function. The variation of *n*-directional vector $Q_0 = [\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n]$ at $\mathbf{x}(t)$ is

$$\dot{Q} = \mathbf{J}_{\mathbf{f}\lambda} Q \tag{3}$$

where

$$\mathbf{J}_{\mathbf{f}\lambda} = \frac{\partial \mathbf{f}_\lambda(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{T}}}\Big|_{\mathbf{x}=\mathbf{x}(t)} \tag{4}$$

is the Jacobian matrix of $\mathbf{f}_\lambda$ with respect to the trajectory under consideration. To $t = T_{\max}$ integral terminal value Q_{end} matrix *QR* decomposition, diagonal value of the upper triangular matrix *R* is

$$|\lambda_1|,|\lambda_2|,\ldots,|\lambda_n|. \quad (5)$$

In order to reduce error accumulation, every once in a while $T_s = T_{\max}/N$ to recalculate Q_0 for units of the decomposed matrix Q (The new direction of the unit).

The spectrum of the Lyapunov exponents l_j is

$$l_j = \lim_{N\to\infty} \frac{1}{NT_s} \sum_{i=0}^{N} \ln|\lambda_j^i| \quad (6)$$

2.2 Calculating pulse switched compensation matrix

Subsystem from one switch to another in switching manifold. Integral results Q_{end} is exclusive switching information. Traditional Lyapunov exponent algorithm methods cannot be applied to impulsive switched systems.

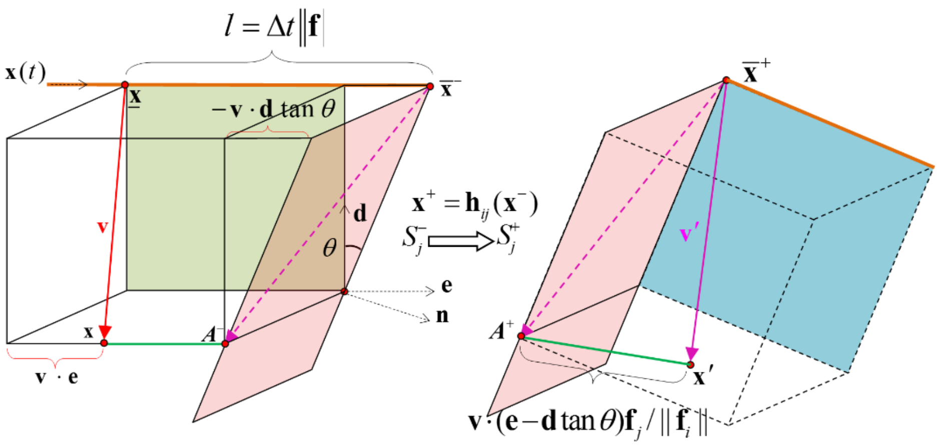


Fig.1 Lyapunov exponent Algorithm derivation of impulsive switched system

Obviously, Jacobian matrix $\mathbf{J}_s$ can be used as a compensation matrix of switching manifold. It is

$$\mathbf{J}_s = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}^T} = \frac{\partial \mathbf{v}'}{\partial \mathbf{v}^T}. \quad (7)$$

As is shown in Fig.1, $\mathbf{e} = \mathbf{f}_i(\bar{x}^-)/\|\mathbf{f}_i(\bar{x}^-)\|$ is the direction of $\mathbf{x}_i(t)$. $\boldsymbol{l} = \Delta \boldsymbol{t}\|\mathbf{f}_i(\underline{\mathbf{x}})\|$ is the length of $\underline{x}$ after a sufficiently small time Δt to $\bar{x}^-$. Where $\bar{x}^-$ is intersection point of orbit with switched section S^-. $\mathbf{n}$ is the normal vector of S^-. The time interval needed for orbit from x to A is

$$\Delta t_1 = \frac{\|\overrightarrow{xA^-}\|}{\|\mathbf{f}_1\|} = \Delta t - \frac{(-\mathbf{v}\cdot\mathbf{d}\tan\theta)+\mathbf{v}\cdot\mathbf{e}}{\|\mathbf{f}_1\|} \tag{8}$$

where is the time interval of $x \to x'$. Ulteriorly, we get

$$\begin{aligned}\overrightarrow{A^+x'} &= \mathbf{f}_2(\Delta t - \Delta t_1) = (\mathbf{v}\cdot\mathbf{e} - \mathbf{v}\cdot\mathbf{d}\tan\theta)\mathbf{f}_2/\|\mathbf{f}_1\| \\ \overrightarrow{x^-A^-} &= [\mathbf{v} - (\mathbf{v}\cdot\mathbf{e}\text{-}\mathbf{v}\cdot\mathbf{d}\tan\theta)\mathbf{e}]\end{aligned}. \tag{9}$$

To obtain the relationship between $\mathbf{v}'$ and $\mathbf{v}$ we should get the relationship between the $\overrightarrow{x^-A^-}$ and $\overrightarrow{x^+A^+}$ before. Then we get the equation as follows:

$$\begin{aligned}\mathbf{v}' &= \overrightarrow{x^+A^+} + \overrightarrow{A^+x'} \\ &= \mathbf{J_h}[\mathbf{v} - (\mathbf{v}\cdot\mathbf{e} - \mathbf{v}\cdot\mathbf{d}\tan\theta)\cdot\mathbf{e}] + (\mathbf{v}\cdot\mathbf{e} - \mathbf{v}\cdot\mathbf{d}\tan\theta)\mathbf{f}_2/\|\mathbf{f}_1\| \\ &= \mathbf{J_h}\mathbf{v} + \mathbf{v}\cdot(\mathbf{e} - \mathbf{d}\tan\theta)(\mathbf{f}_2/\|\mathbf{f}_1\| - \mathbf{J_h}\mathbf{e})\end{aligned} \tag{10}$$

Thus we obtain compensation Jacobian matrix of the pulse is:

$$\mathbf{J_s} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}^T} = \frac{\partial \mathbf{v}'}{\partial \mathbf{v}^T} = \mathbf{J_h} + (\mathbf{f}_2/\|\mathbf{f}_1\| - \mathbf{J_h}\mathbf{e})(\mathbf{e} - \tan\theta\,\mathbf{d})^T \tag{11}$$

3. Example: the simplest dynamic walking model

The simplest walking model of a typical pulse switch system is the basis of the theory of dynamic walking robot. Research Lyapunov exponent calculation of dynamic walking is very significant, especially the gait analysis and judgement. This model as an application example below demonstration the implementation process of the algorithm in this paper.

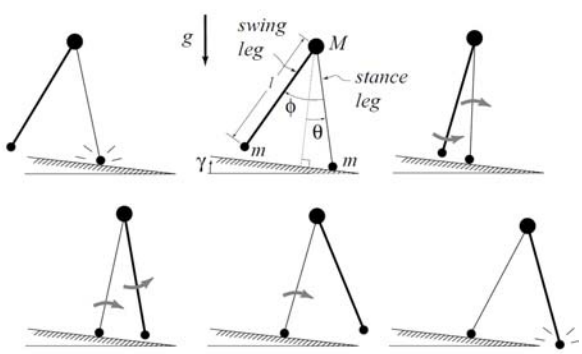
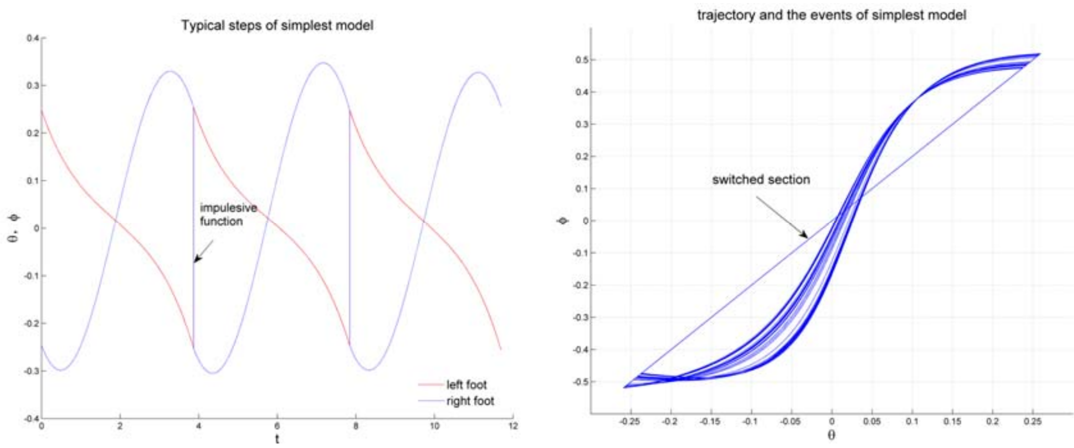


Fig.2 A typical passive walking step.

The new stance leg has just made contact with the ramp in the upper left picture. The swing leg swings until the next heel-strike[10]. The top-center picture gives a description of the variables and parameters that we use. θ is the angle of the stance leg with respect to the slope normal. ϕ is the angle between the stance leg and the swing leg. M is the hip mass, and m is the foot mass. l is the leg length. γ is the ramp slope, and g is the acceleration due to gravity. Reprinted with permission from Garcia et al[11].



(a) (b)

Fig.3 Typical phase of simplest model and its switched section

The system simulation results as shown in fig.3. Where red line shows the state of support legs, blue line shows the same of swinging leg and vertical blue lines represent role switching under the impulsive function in subplot (a). We can see system switching when touch switched section from subplot (b).

Use Poincaré mapping measure will reduce 2-dimensional of simplest system and get two Lyapunov exponents (0.0966, -0.4250). But 4-dimensional chaotic system[23] will contain four exponents and one of them must be zero.

Now we use the algorithm in this paper to calculate Lyapunov exponents of the system. Got direction vector $\mathbf{n}=[-2,1,0,0]^{\mathrm{T}}$ by impulsive function h. According to $\mathbf{e}=f(\mathrm{x})/\|f(\mathrm{x})\|$ we get equation as

$$\begin{gathered}\theta=\arccos\mathbf{n}\cdot\mathbf{e}\\ \mathbf{d}=\frac{(\mathbf{n}\cdot\mathbf{e})\mathbf{e}-\mathbf{n}}{\|(\mathbf{n}\cdot\mathbf{e})\mathbf{e}-\mathbf{n}\|}\end{gathered} \quad . \tag{12}$$

Calculating the Lyapunov exponent of simplest model without $\mathbf{J}_{\mathrm{s}}$ is (0.9963, 0.0000, -0.0003, -0.9960) [24]. Then use the compensatory Jacobian to modify the pulse and get the result is (0.0954 0.0001, -0.4238, -Inf). Obviously the first and third exponent compare with traditional Jacobian algorithm obtained results consistent within error scope. These numbers prove that this system is a chaotic system which consistent with the fact. Thus verify the correctness of this paper's method through the simplest model.

4. Conclusion

The model based algorithm for the calculation of the spectrum of Lyapunov exponents has been generalized for the case of nonlinear dynamical systems with discontinuities. The main result is the supplementation of the continuous subsystems of a certain compensation matrix at the instants of discontinuities. For example of simplest dynamic walking model with 4-dimensional we calculated out all Lyapunov exponents and the result support our algorithm.

Even the generalized algorithm is straightforward but there are still open problems in its efficient implementation. The main problem is precision of Jacobian matrix. With the increase of the integral number, the error will be magnified. Fortunately, this effects within the precision of the system error. But this problem is still under consideration.

Acknowledgements

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