Journal of Computing and Electronic Information Management

ISSN: 2413-1660

Application of Deflated SPIKE Recursion on Two-Dimensional Single-Phase A3-Grid Seepage Equation

Shiquan Li

College of Science, University of Shanghai for Science and Technology, Shanghai, China

Abstract: A3-grid arrangement is commonly used in reservoir numerical simulation. Based on the particularity of its coefficient matrix structure, a more efficient new algorithm can be constructed to solve the system. First, for traditional ILU preconditioners, the error is too large. Only through the SPIKE process to reduce the dimension of the original large matrix can the computational complexity be reduced and the accuracy of the original solution be kept. After special adjustments, it can be found that the system can accept a recursive SPIKE process.

Keywords: Seepage model; A3-grid arrangement; Deflated SPIKE recursive algorithm.

1. Introduction

In reservoir numerical simulation, differential equations are usually discretized and linearized to achieve a large complex linear system. However, the number of phases, the way of arrangement, discrete format and other processing methods will affect the form of linear system and iterative performance and complexity of the algorithm. There is no effective targeted algorithm to deal with such systems. The traditional algorithm [1] tries to preserve the upper and lower triangular arrays of the original non-zero position through the ILU(0) (0-level-filled incomplete LU decomposition [2]) preconditioner, and then through ORTHOMIN algorithm.

So far, the methods for solving sparse linear algebraic equations are usually divided into direct and iterative methods. The direct method is generally based on sparse triangular decomposition techniques, including LU decomposition, QR decomposition, etc., and the subspace iteration method of the generalized minimal residual method (GMRES) in the iterative method is the most representative. Compared with the direct method, iterative methods have the advantages of less storage. However, the convergence of the iterative method strongly depends on the spectral distribution of

the coefficient matrix, so scientists are committed to exploring preconditioners [7,9,11,12]. Weisi Zhou [3] uses IC-CG to give a new preconditioner iterative algorithm. Compared with the traditional ILU(0)-GMRES, solving the symmetric matrix should have a faster convergence speed. Peng Dou [4] has innovated the ILU and defined the staged preconditioner Hybrid-LU, which uses the precise and inexact decomposition to deal with the algebraic system.

Based on the results of the above research, improvements in structural direction will continue steadily [5,10]. In this paper, a new structural recursive algorithm is presented to solve the system under the A3-grid arrangement.

2. Seepage Model

The mathematical model describing the fluid seepage in a reservoir is generally:

$$\nabla\left[\frac{\rho k k_r}{\mu}\left(\nabla P-\rho g D\right)\right]+q=\frac{\partial}{\partial t}\left(\phi\rho s\right)$$

where ρ refers to the density of the fluid; k refers to permeability, an ability of fluid to pass the rock under a pressure; when there is material exchange between multiple fluids, k_r is the relative permeability; μ is the viscosity of the fluid; q is the sink source term, representing the input and output of the fluid well; ϕ is porosity, which is the physical parameter of rock.

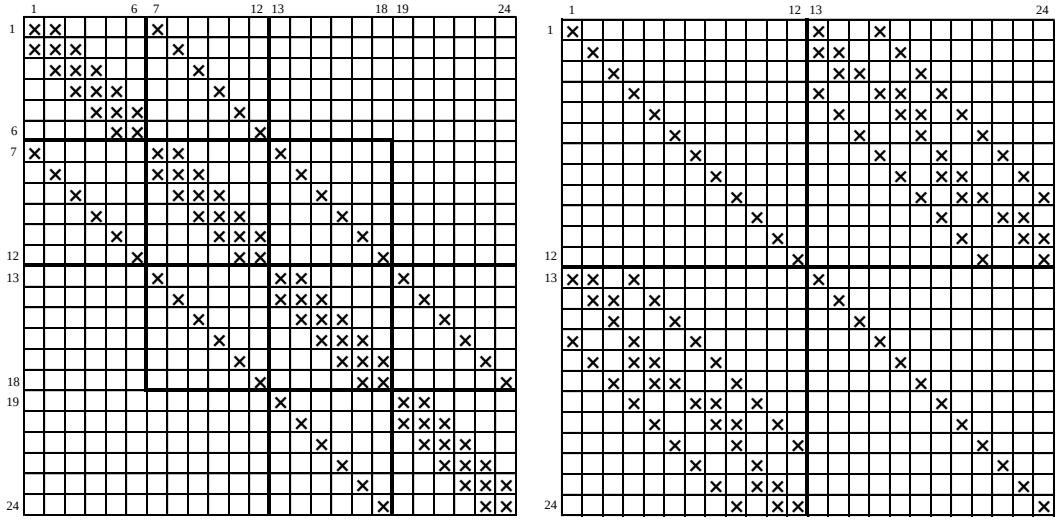
The unknowns of the model are pressure P and saturation s, which are both a function of space and time, denotes $P(x,y,t)$ and $s(x,y,t)$. In the discrete process, through IMPES, whether the semi-implicit format or the fully implicit format is re-linearized, a system of equations $AP=b$ containing only the unknown vector P can be obtained, and then the original equation s will be solved by the back-generation equation. This paper considers the single-phase flow problem, so the saturation s takes 1 directly. As an example, dividing the average 6 meshes in the x direction and dividing the 4 meshes in the y direction in the solution area. A penta-diagonal coefficient matrix with empty gap can be obtained according to the standard arrangement format, shown in Fig.1(c). Three diagonal lines are closely arranged, after separating the half bandwidth, there are two outer diagonal lines. When adopting the A3-grid arrangement format, the structure is particularly clear, shown in Fig.1(d).

y=4	19	20	21	22	23	24
3	13	14	15	16	17	18
2	7	8	9	10	11	12
1	1	2	3	4	5	6
x=	1	2	3	4	5	6

(a)

y=4	15	5	19	9	23	12
3	2	16	6	20	10	24
2	13	3	17	7	21	11
1	1	14	4	18	8	22
x=	1	2	3	4	5	6

(b)



(c) (d)

Fig. 1 Standard arrangement/A3 arrangement and the non-zero position of their corresponding matrix

Considering the following *ab* × *ab* linear system $Ax = b$ (in the *x* direction *a* split, *y* direction *b* split), matrix A shaped like

$$\mathcal{A} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array}\right] \in \mathbb{R}^{ab\times ab}$$

Recording that *ab=2n*, then $\mathbf{A}, \mathbf{D} \in \mathbb{R}^{n\times n}$ is a diagonal and non-singular matrix, and $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ is a loose penta-diagonal matrix of the following format, which can also be turned into a tridiagonal block matrix, shown in Fig.2

$$\begin{bmatrix} B_{11} & B_{12} & & & \\ B_{21} & B_{22} & B_{23} & & \\ & B_{32} & B_{33} & B_{34} & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \text{ where } \mathrm{B_{ij}} \in \mathbb{R}^{3\times 3}$$

Fig. 2 Rewriting matrix B in tridiagonal block form

Facing the high-order linear system, we first reduce the order by Schur Complement.

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \xrightarrow{r_2 - \mathbf{C}\mathbf{A}^{-1}r_1} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \xrightarrow{r_2 - \mathbf{C}\mathbf{A}^{-1}r_1} \begin{bmatrix} b_1 \\ b_2 - \mathbf{C}\mathbf{A}^{-1}b_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \end{bmatrix}$$

After that, the 2*n*-th order equation can be solved into two *n*-th order subsystems

$$\begin{cases} \mathbf{A}x_1 + \mathbf{B}x_2 = b_1 \\ \mathbf{D}'x_2 = b_2' \end{cases}$$

Obviously, the shape of $\mathbf{D}'$ becomes a loose nine-diagonal matrix. The preconditioned method commonly used in reservoir equations is ILU(0), that means, the original non-

zero position matrix will be reserved by the equation $A = \hat{L}\hat{U} + E$; For this nine-diagonal matrix, which can also be written as a structure whose tridiagonal block is dense, and the two side blocks are diagonal blocks; Therefore, when the grid division is finer and the matrix is larger, the non-zero-element positions that need to be zeroed in the ILU decomposition are also suddenly increased, and all are classified into the indicator array *E*. There is no doubt that this split will lose a lot of information, and as time progresses, the error will get bigger and bigger.

3. Deflated SPIKE Recursive Process

3.1 Standard SPIKE Process

The banded system can be divided into three diagonal forms [6], if the diagonal blocks can be equally divided into *p*, then the order of each diagonal block $A_j(j = 1, \dots, p)$ is $n_p = \frac{n}{p}$.

Under the same segmentation, B_j and C_j are the tiny lower triangular array and the tiny upper triangular array adjacent to the main diagonal block. Their order is denoted as *m*, obviously $m \ll n_p$. How to divide *m* and n_p is free, depending on the specific system structure. In general, the system $Tf=s$ is divided into *p* layers, and *f*、*s* can be split into f_j、s_j accordingly, shown in Fig.3

Assuming that each A_j diagonal block is non-singular, the matrix can be decomposed into $T = D_T S_T$, where D_T matrix is the diagonal block matrix of matrix ***T***. D_T =diag($A_1,\dots,A_p$)

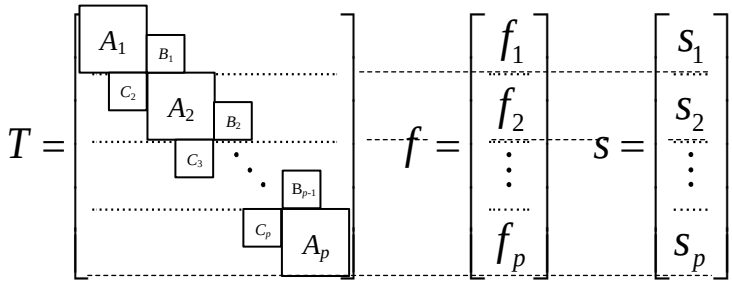


Fig. 3 p layers of special tridiagonal matrix

Fig.4 shows the corresponding ST matrix and solution vector adopt the same division. Under the same division, V_j and W_j correspond to the accretive rectangular matrix on the left and right sides of the diagonal block, and the new RHS is recorded as d_j.

$$V_j = \left(A_j\right)^{-1}\begin{bmatrix}0\\ I_m\end{bmatrix}B_j \quad W_j = \left(A_j\right)^{-1}\begin{bmatrix}I_m\\ 0\end{bmatrix}C_j \quad d_j = \left(A_j\right)^{-1}s_j$$

At this point, the system $Tx=s$ is turned into a new solving system $S_T f=d$. For the *p* row in the S_T matrix, continue to subdivide into the upper *m* rows, the middle *np-2m* rows, and the lower *m* rows. A more visual interpretation of the division principle,

shown as Fig.5

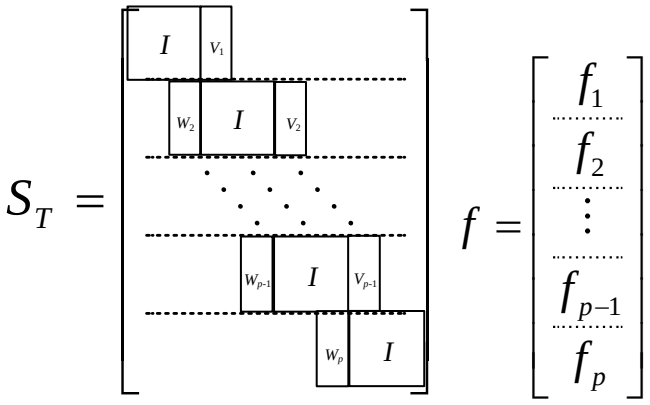


Fig. 4 Rewriting the initial system in the same p layer

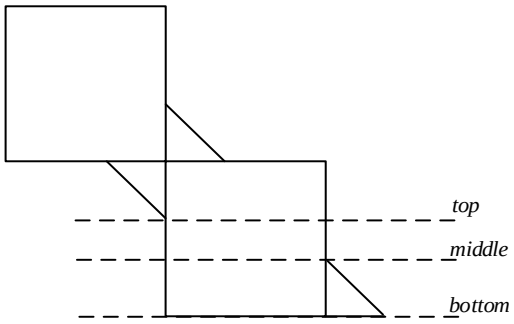


Fig. 5 Establishing SPIKE partition based on the shape of the original matrix

Meanwhile, the corresponding accretion matrix, solution vector, and RHS in the system are all divided into top/middle/bottom layer.

$$V_j = \begin{bmatrix} V_j^{(t)} \\ V_j^{(m)} \\ V_j^{(b)} \end{bmatrix} W_j = \begin{bmatrix} W_j^{(t)} \\ W_j^{(m)} \\ W_j^{(b)} \end{bmatrix} f_j = \begin{bmatrix} f_j^{(t)} \\ f_j^{(m)} \\ f_j^{(b)} \end{bmatrix} d_j = \begin{bmatrix} d_j^{(t)} \\ d_j^{(m)} \\ d_j^{(b)} \end{bmatrix}$$

After the second subdivision, a reduced order subsystem $\hat{S}\hat{f} = \hat{d}$ can be extracted. This reduced system is block tridiagonal with (p-1) diagonal blocks, the k-th of which is given by $\begin{bmatrix} I_m & V_k^{(b)} \\ W_{k+1}^{(t)} & I_m \end{bmatrix} \in \mathbb{R}^{2m\times 2m}$, non-diagonal blocks adjacent to the left and right sides are given by $\begin{bmatrix} W_{k+1}^{(b)} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2m\times 2m}$ and $\begin{bmatrix} 0 & 0 \\ 0 & V_{k+1}^{(t)} \end{bmatrix} \in \mathbb{R}^{2m\times 2m}$.

The simplified system of fine segmentation reduces the dimension from n-th order to 2m(p-1)-th order, and the solution vector of the reduced order system can be quickly obtained. After acquisition, the solution of the complete system can be reconstructed by the following relationship.

$$\begin{cases} f_1^{(m)} = d_1^{(m)} - V_1^{(m)} f_2^{(t)} \\ f_j^{(m)} = d_j^{(m)} - V_j^{(m)} f_{j+1}^{(t)} - W_j^{(m)} f_{j-1}^{(b)} \\ f_p^{(m)} = d_p^{(m)} - W_p^{(m)} f_{p-1}^{(b)} \end{cases}$$

3.2 Deflated SPIKE Process applied to A3-grid arrangement

Recalling the case that the structure of the linear system is got after the coefficient matrix extracted by the A3-grid arrangement through Schur Complement process. No matter how the system is divided, the adjacent matrix shapes on both sides are invariant, shown as Fig.6

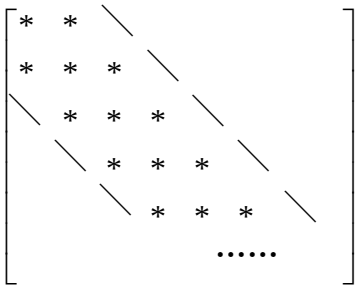


Fig. 6 the matrix structure after SC process in A3-grid

Where " * " means dense block matrix, "＼" means diagonal block matrix. Whether cutting from the edge of the block or from the inside of the block, the order of matrix B_j (also C_j) is twice the order of standard block matrix as the structure shows.

Facing this special arrangement, conditions n_p=2m can be determined directly (shown as Fig.7) so that segmentation of the middle layer can be avoided, and it destined to increase the complexity of the reduced system solution for its slightly higher order.

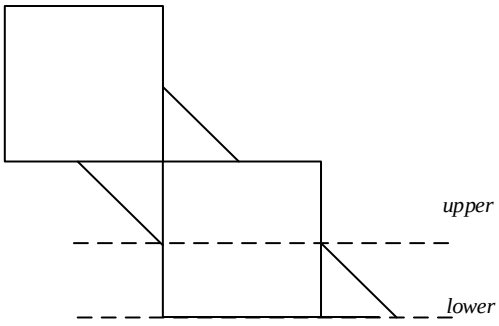


Fig. 7 Deflated SPIKE partition

Applying the original SPIKE process according to the new division, the new subsystem was successfully extracted, written in the form of matrix-vector product. It is not difficult to find that the matrix can be column transformed and reconstructed into a block tridiagonal form, shown as Fig.8

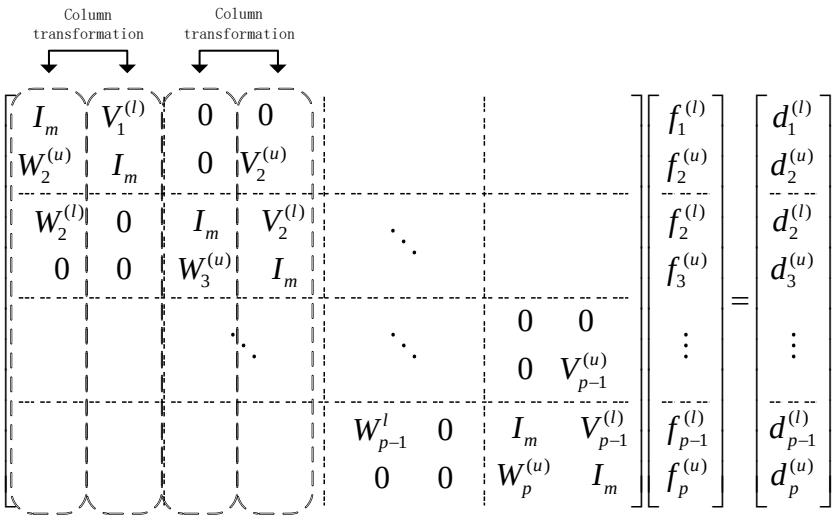


Fig. 8 Extracting the reduction matrix

Simultaneously multipling the permutation matrix P on both sides of the equation as a preconditioner, and this exchange has completed clearly. The value of P is easy to know, because the affection is that each couple adjacent column is interchanged, so P is the matrix after interchange of each couple adjacent column of the unit matrix.

$$APu = d,\ f = Pu$$

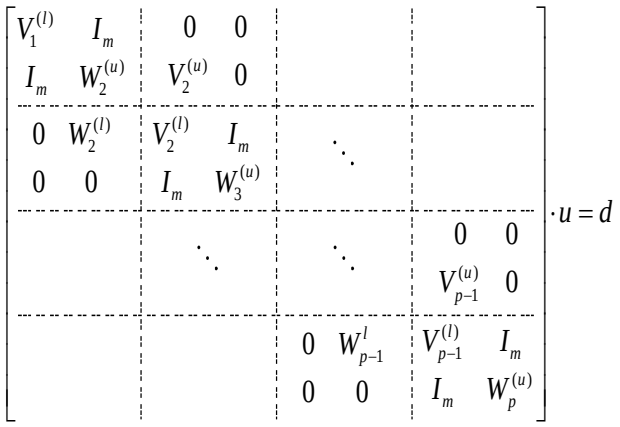


Fig. 9 After column transformation

Once again, a matrix of three diagonal blocks with large main block and small block on both sides (shown as Fig.9) is obtained. Recursively, deflated SPIKE process can be performed again. The following will make a standard code for the recursive system, shown as Fig.10

The initial SPIKE process extracts the square matrix of $2m(p\text{-}1)$-th order, and then, after each SPIKE process, the dimension is reduced by $2m$. In principle, at most p-1 deflated SPIKE processes can be performed, and finally a 2×2 block equation is obtained.

$$\begin{bmatrix} I_m & V_1^{\{p-1\},(l)} \\ W_2^{\{p-1\},(u)} & I_m \end{bmatrix} \cdot \hat{u}_{p-3} = \hat{d}_{p-2}$$

The superscript $\{t\}$ records the recursion times. The relationship of the solution vector between the upper system and its subsystem will be stored with the matrix, and finally regressed

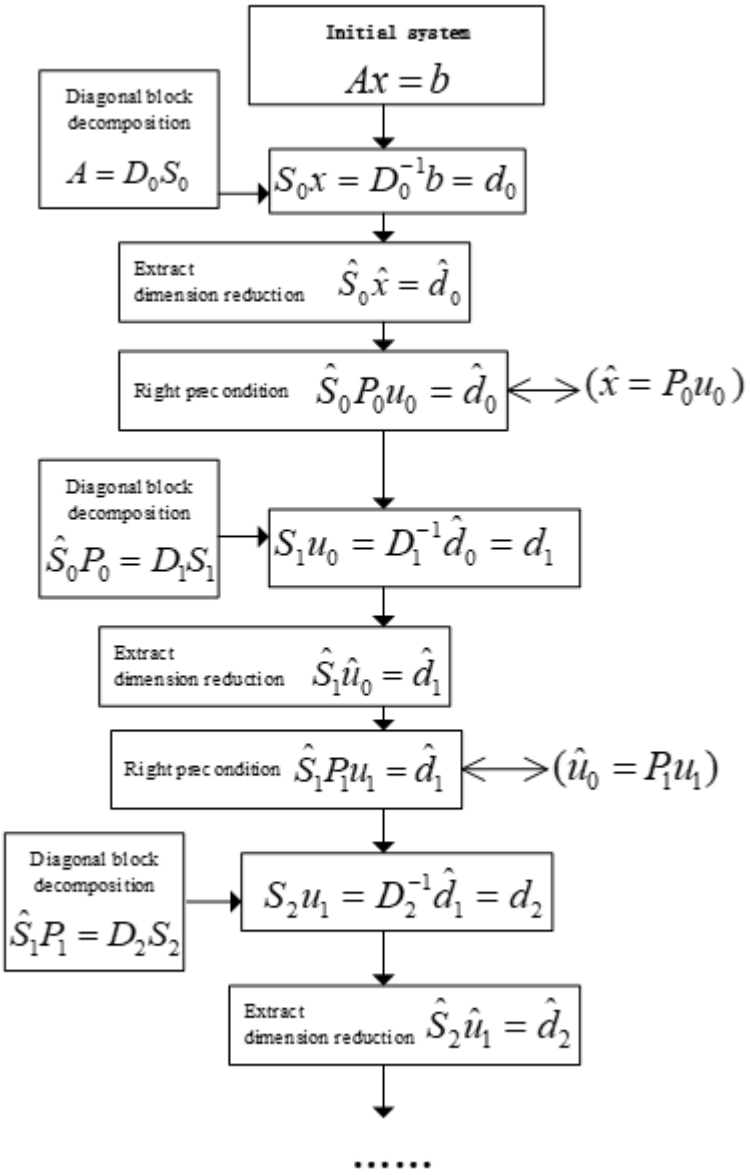


Fig. 10 Standard code for the SPIKE recursive

4. Conclusion

The Deflated SPIKE recursive algorithm is an iterative process on the structure, not a numerical solution process. When faced with a large complex algebraic system, the exact numerical iterations will become extremely slow, because the memory used will

be large as the number of iterations is accumulated; If you use the inaccurate optimizations, indeed, it is fast and relatively accurate to deal with small, sparse algebraic systems. But once the order of the matrix is large, the value of the rounded iteration will be more and more deviated from the analytical solution.

The Deflated SPIKE recursive process can quickly reduce the order matrix while maintaining the accuracy of the solution. The correspondence between the upper level system solutions and its subsystem solutions is also very clear; Of course, the algorithm also has some drawbacks. If there are too many layers, the equations that need to be stored will also become a lot, which will take up much memory.

Acknowledgements

This paper was financially supported by Key Discipline Construction Funding Project of Shanghai, project number: B14005

References

[1] Shuxia Li and Jianwei Gu, "Basic Numerical Simulation of Reservoirs", China University of Petroleum Press, 2009, p127-131

[2] Yousef Saad, "Iterative methods for sparse linear systems", Society for Industrial and Applied Mathematics, 2003, p269-280

[3] Weisi Zhou, "Incomplete Cholesky Decomposition of Two-dimensional Two-phase Model——Conjugate Slant Solution Method", Journal of Petroleum, 1986, Vol. 7(2), p83-90

[4] Peng Dou and Ming Ma, "Parallel Algorithm for Transient Stability Simulation Based on GMRES Method Preconditioned by Hybrid-LU Decomposition", High Voltage Engineering, 2017, Vol. 43(10), p3419-3426

[5] Garey LE and Shaw RE, "Solving Banded and Near Symmetric Systems", Applied Mathematics and Computation, 2000, Vol. 115, p133-143

[6] Eric Polizzi and Ahmed H Sameh, "A Parallel Hybrid Banded System Solver: the SPIKE Algorithm", Parallel Computing, 2006, Vol. 32, p177–194

[7] J.R. Wallis, "Incomplete Gaussian Elimination as a Preconditioning for Generalized Conjugate Gradient Acceleration", Society of Petroleum Engineers of AIME, 1983, SPE 12265

[8] Andrew Chapman and Yousef Saad, "Deflated and Augmented Krylov Subspace Techniques", Numerical Linear Algebra with Applications, 1997, Vol.4(1), p43-66

[9] Haiyan Zhu, "ILU Decomposition Block Iterative Method for Solving Interval Linear Equations", Journal of Shandong Polytechnic University, 1993, Vol. 23(2), p71-77

[10] Xinrong Ma, Sanyang Liu and Zhijian Duan, "Parallel Alternating Direction Algorithm with Parameters for Solving Banded Linear Systems", Computer Science, 2014, Vol.41(2), p249-252

[11] Min Dai, Yanling Guo and etc., "Implementation of the Relaxed ILU Preconditioner to Reservoir Simulators", Numerical Calculation and Computer Application, 2004, Vol.3,

p183-191
[12] Jimin Shi and Zhenbao Lin, "MILU-CG of Three-dimensional Three-phase Black Oil Model Comparison of Methods and Numerical Results", Petroleum Exploration and Development, 1994, Vol.21(6), p46-52